

Dynamical Instability and Expansion-free Condition in $f(R, T)$ Gravity

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Abstract

Dynamical analysis of spherically symmetric collapsing star surrounding in locally anisotropic environment with expansion-free condition is presented in $f(R, T)$ gravity, where R corresponds to Ricci scalar and T stands for the trace of energy momentum tensor. The modified field equations and evolution equations are reconstructed in the framework of $f(R, T)$ gravity. In order to acquire the collapse equation we implement the perturbation on all matter variables and dark source components comprising the viable $f(R, T)$ model. The instability range is described in Newtonian and post-Newtonian approximation. It is observed that the unequal stresses and density profile defines instability range rather than the adiabatic index. However, the physical quantities are constrained to maintain positivity of energy density and stable stellar configuration.

Keywords: Collapse; $f(R, T)$ gravity; Dynamical equations; Instability range; Expansion-free condition.

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1 Introduction

The astrophysics and astronomical theories are invigorated largely by the gravitational collapse and instability range explorations of self gravitating objects. Celestial objects tend to collapse when they exhaust all their nuclear fuel, gravity takes over as the inward governing force. The gravitating bodies undergoing collapse face contraction to a point that results in high energy dissipation in the form of heat flux or radiation transport [1]. The end state of stellar collapse has been studied extensively, a continual evolution of compact object might end up as a naked singularity or as black hole depending upon the size of collapsing star and also on the background that plays important role in pressure to gravity imbalances [2]-[4].

The gravitating objects are interesting only when they are stable against fluctuations, supermassive stars tends to be more unstable in comparison to the less massive stars [5]. Instability problem in star's evolution is of fundamental importance, Chandrasekhar [6] presented the primary explorations on dynamical instability of spherical stars. He identified instability range of star having mass M and radius r by a factor Γ pertaining the inequality $\Gamma \geq \frac{4}{3} + n\frac{M}{r}$. Adiabatic index measures compressibility of the fluid i.e., variation of pressure with a given change in the density. The analysis of expanding and collapsing regions in gravitational collapse was presented by Sharif and Abbas [7].

Herrera and his collaborators [8]-[11] presented the dynamical analysis associated with isotropy, local anisotropy, shear, radiation and dissipation with the help of Γ , it was established that minor alterations from isotropic profile or slight change in shearing effects bring drastic changes in range of instability. However, instability range of stars with zero expansion does not depend on stiffness of fluid, rather on other physical parameters [12]-[14], such as mass distribution, energy density profile, radial and tangential pressure. The impact of local anisotropy on plane expansion-free gravitational collapse is studied in [15].

General Relativity (GR) facilitates in providing field equations that leads to the dynamics of universe in accordance with its material ingredients. The predictions of GR are suitable for small distances, however, there are some limitations of GR in description of late time universe. Modified gravity theories have been widely used to incorporate dark energy components of universe by inducing alterations in Einstein Hilbert (EH) action. Due to modifications in laws of gravity at long distances, dark source terms of modified gravity

leaves phenomenal observational signatures such as cosmic microwave background, weak lensing and galaxy clustering [16]-[21]. Many people investigated the dynamics of collapse and instability range in modified theories of gravity, Cembranos et al. [22] studied the collapse of self-gravitating dust particles. Sharif and Rani [23] established the instability range of locally anisotropic non-dissipative evolution in $f(T)$ theory.

Among modified gravity theories, $f(R)$ exhibits the most elementary modifications to EH action by adopting a general function $f(R)$ of Ricci scalar. Ghosh and Maharaj [24] indicated that null dust non-static collapse in $f(R)$ around de-Sitter higher dimensional background leads to naked singularity. Combined effect of electromagnetic field and viable $f(R)$ model has been investigated in [25], concluding that inclusion of Maxwell source tends to enhance the stability range. Borisov et al. [26] investigated the spherically symmetric collapse of $f(R)$ models with non-linear coupling scalar by execution of one-dimensional numerical simulations. The dynamical instability of extremal Schwarzschild de-Sitter background framed in $f(R)$ is investigated in [27].

Another modification of GR and generalization of $f(R)$ was presented in 2011 by Harko et al. [28] termed as $f(R, T)$ gravity theory constituting the matter and geometry coupling, EH action is modified in a way that gravitational Lagrangian includes higher order curvature terms alongwith the trace of energy momentum tensor T . Shabani and Farhoudi [29] explained the weak field limit by applying dynamical system approach and analyzed the cosmological implications of $f(R, T)$ models with a variety of cosmological parameters such as Hubble parameter, its inverse, snap parameters, weight function, deceleration, jerk and equation of state parameter. Ayuso et al. [30] worked on consistency criterion for non-minimally coupled class of modified theories of gravity. Sharif and Zubair [31]- [34] ascertained the laws of thermodynamics, energy conditions and analyzed the anisotropic universe models in $f(R, T)$ framework.

Chakraborty [35] explored various aspects of homogeneous and isotropic cosmological models in $f(R, T)$ and formulate the energy conditions for perfect fluid. Dynamics of scalar perturbations in $f(R, T)$ is explored in [36]. Jamil et al. [37, 38] reconstruct some cosmological models and studied laws of thermodynamics in $f(R, T)$. In a recent paper [39], dynamical instability of isotropic collapsing fluid in the context of $f(R, T)$ is considered. We have also discussed the stability analysis of spherically symmetric collapsing star surrounding in locally anisotropic environment in $f(R, T)$ gravity [40]. Fur-

thermore, conditions on physical quantities are constructed for Newtonian and post-Newtonian eras to address instability problem.

Herein, we intend to develop the instability range of $f(R, T)$ model under anisotropic background constraining to zero expansion. The expansion-free condition necessarily implies the appearance of cavity within fluid distribution that might help in modeling of voids at cosmological scales. Also, such distributions must bear energy density inhomogeneities that are incorporated here by inducing non-constant energy density and pressure anisotropy. The dynamical analysis of various fluid distributions with expansion-free condition has been studied in $f(R)$ [41]-[43], however, such situations have not been covered yet in $f(R, T)$. Recently, Noureen and Zubair [44] discussed the implications of extended Starobinsky Model on dynamical instability of axially symmetric gravitating body.

To develop collapse equation in $f(R, T)$, we construct corresponding field equations constituting expansion-free fluid. The action in $f(R, T)$ is as in [28]

$$\int dx^4 \sqrt{-g} \left[\frac{f(R, T)}{16\pi G} + \mathcal{L}_{(m)} \right], \quad (1.1)$$

where $\mathcal{L}_{(m)}$ is matter Lagrangian and g denotes the metric tensor. The Lagrangian $\mathcal{L}_{(m)}$ can assume various choices, each choice corresponds to a set of field equations for some special form of fluid. Here, we have chosen $\mathcal{L}_{(m)} = \rho$, $8\pi G = 1$ and upon variation of above action with metric g_{uv} the field equations are formed as

$$\begin{aligned} G_{uv} = & \frac{1}{f_R} \left[(f_T + 1) T_{uv}^{(m)} - \rho g_{uv} f_T + \frac{f - R f_R}{2} g_{uv} \right. \\ & \left. + (\nabla_u \nabla_v - g_{uv} \square) f_R \right], \end{aligned} \quad (1.2)$$

where $T_{uv}^{(m)}$ denotes the energy momentum tensor for usual matter.

The matter Lagrangian is configured in a way that it depends only on the components of metric tensor [45]. In order to present the dynamical analysis we implement the linear perturbation on collapse equation, assuming that initially all physical quantities are in static equilibrium. The paper is arranged as: Einstein's field equations and dynamical equations for $f(R, T)$ are constructed in section 2 that leads to the collapse equation. In section 3 perturbation scheme is implemented to the dynamical equations. Section 4 covers the discussion of expansion-free condition and the components affecting the stability of gravitating objects, extracted from perturbed Bianchi

identities along with corrections to Newtonian and post-Newtonian eras and GR solution. Section 5 comprises the summary followed by an appendix.

2 Dynamical Equations in $f(R, T)$

We choose a three dimensional external spherical boundary surface Σ that pertains two regions of spacetime termed as interior and exterior regions. The line element for region inside the boundary Σ is of the form

$$ds_-^2 = W^2(t, r)dt^2 - X^2(t, r)dr^2 - Y^2(t, r)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.3)$$

The domain beyond (lying outside) Σ is exterior region with following line element [43]

$$ds_+^2 = \left(1 - \frac{2M}{r}\right) d\nu^2 + 2drd\nu - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.4)$$

where ν is the corresponding retarded time and M is total mass. To arrive at onset of field equations given in Eq.(1.2), we choose $T_{uv}^{(m)}$ describing anisotropic fluid distribution of usual matter, given as

$$T_{uv}^{(m)} = (\rho + p_\perp)V_u V_v - p_\perp g_{uv} + (p_r - p_\perp)\chi_u \chi_v, \quad (2.5)$$

where ρ denotes energy density, V_u is four-velocity of the fluid, χ_u corresponds to radial four vector, p_r and p_\perp represent the radial and tangential pressures, respectively. The physical quantities appear in energy momentum tensor are in accordance with the following identities

$$V^u = W^{-1}\delta_0^u, \quad V^u V_u = 1, \quad \chi^u = X^{-1}\delta_1^u, \quad \chi^u \chi_u = -1. \quad (2.6)$$

The expansion scalar Θ defines rate of change of small volumes of the fluid, given by

$$\Theta = V_{;u}^u = \frac{1}{W} \left(\frac{\dot{X}}{X} + 2 \frac{\dot{Y}}{Y} \right), \quad (2.7)$$

where dot and prime denote the time and radial derivatives respectively. The components of field equations for spherically symmetric interior spacetime are

of the form

$$G_{00} = \frac{1}{f_R} \left[\rho + \frac{f - Rf_R}{2} + \frac{f_R''}{X^2} - \frac{\dot{f}_R}{W^2} \left(\frac{\dot{X}}{X} + \frac{2\dot{Y}}{Y} \right) - \frac{f_R'}{X^2} \left(\frac{X'}{X} - \frac{2Y'}{Y} \right) \right], \quad (2.8)$$

$$G_{01} = \frac{1}{f_R} \left[\dot{f}_R' - \frac{W'}{W} \dot{f}_R - \frac{\dot{X}}{X} f_R' \right], \quad (2.9)$$

$$G_{11} = \frac{1}{f_R} \left[p_r + (\rho + p_r) f_T - \frac{f - Rf_R}{2} + \frac{\ddot{f}_R}{W^2} - \frac{\dot{f}_R}{W^2} \left(\frac{\dot{W}}{W} - \frac{2\dot{Y}}{Y} \right) - \frac{f_R'}{X^2} \left(\frac{W'}{W} + \frac{2Y'}{Y} \right) \right], \quad (2.10)$$

$$G_{22} = \frac{1}{f_R} \left[p_\perp + (\rho + p_\perp) f_T - \frac{f - Rf_R}{2} + \frac{\ddot{f}_R}{W^2} - \frac{f_R''}{X^2} - \frac{\dot{f}_R}{W^2} \left(\frac{\dot{W}}{W} - \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} \right) - \frac{f_R'}{X^2} \left(\frac{W'}{W} - \frac{X'}{X} + \frac{Y'}{Y} \right) \right]. \quad (2.11)$$

The dynamical equations are important in establishment of the instability range of collapsing stars. Misner and Sharp mass function furnishes the total amount of energy in a spherical star of radius "Y" and facilitates in formulation of dynamical equations, given by [46]

$$m(t, r) = \frac{Y}{2} \left(1 + \frac{\dot{Y}^2}{W^2} - \frac{Y'^2}{B^2} \right), \quad (2.12)$$

The matching conditions of adiabatic sphere on the boundary surface of exterior spacetime results from continuity of differential forms as [47]

$$M \stackrel{\Sigma}{=} m(t, r) \quad (2.13)$$

The dynamical analysis can be established by using conservation laws, we have taken conservation of Einstein tensor because the energy momentum tensor bear non-vanishing divergence in $f(R, T)$ gravity. The contracted Bianchi identities imply dynamical equations that further leads to collapse equation, given by

$$G_{;v}^{uv} V_u = 0, \quad G_{;v}^{uv} \chi_u = 0, \quad (2.14)$$

Bianchi identities in account with the Eq.(2.7) become

$$\begin{aligned} \dot{\rho} + \rho \left\{ [1 + f_T] \Theta - \frac{\dot{f}_R}{f_R} \right\} + [1 + f_T] \left\{ p_r \frac{\dot{X}}{X} + 2p_\perp \frac{\dot{Y}}{Y} \right\} \\ + Z_1(r, t) = 0, \end{aligned} \quad (2.15)$$

$$\begin{aligned} (\rho + p_r) f'_T + (1 + f_T) \left\{ p'_r + \rho \frac{W'}{W} + p_r \left(\frac{W'}{W} + 2 \frac{Y'}{Y} - \frac{f'_R}{f_R} \right) \right. \\ \left. - 2p_\perp \frac{Y'}{Y} \right\} + f_T \left(\rho' - \frac{f'_R}{f_R} \right) + Z_2(r, t) = 0, \end{aligned} \quad (2.16)$$

where $Z_1(r, t)$ and $Z_2(r, t)$ are the corresponding terms including dark matter components provided in **Appendix** as Eqs.(6.1) and (6.2) respectively. These equations are useful in the description of variation from equilibrium to the evolution.

3 $f(R, T)$ Model and Perturbation Scheme

The $f(R, T)$ model we have considered for evolution analysis is

$$f(R, T) = R + \alpha R^2 + \lambda T, \quad (3.17)$$

where α and λ corresponds to the positive real values. Generally, a viable model represents the choice of parameters whose variation shall be in accordance with the observational situations [47]. Astrophysical models are selected by checking their cosmological viability, that must be fulfilled to extract consistent matter domination phase, to assemble solar system tests and stable high-curvature configuration recovering the standard GR. The model under consideration is consistent with the stable stellar configuration because second order derivative with respect to R remains positive for assumed choice of parameters.

The field equations in $f(R, T)$ are highly complicated, their general solution is a heavier task and has not been accomplished yet. Evolution of linear perturbations can always be used to study the gravitational modifications by avoiding such discrepancies. The concerned collapse equation can be furnished by application of linear perturbation on dynamical equations along with the static configuration of field equations leading to the instability range. The dynamical analysis can be anticipated either by following

fixed or co-moving coordinates i.e., Eulerian or Lagrangian approach respectively [47]. Since universe is almost homogeneous at large scale structures that is why we have used co-moving coordinates.

Initially all physical quantities are considered to be in static equilibrium so that with the passage of time these have both the time and radial dependence. Taking $0 < \varepsilon \ll 1$ the perturbed form of quantities along with their initial form can be written as

$$W(t, r) = W_0(r) + \varepsilon D(t)w(r), \quad (3.18)$$

$$X(t, r) = X_0(r) + \varepsilon D(t)x(r), \quad (3.19)$$

$$Y(t, r) = Y_0(r) + \varepsilon D(t)\bar{y}(r), \quad (3.20)$$

$$\rho(t, r) = \rho_0(r) + \varepsilon \bar{\rho}(t, r), \quad (3.21)$$

$$p_r(t, r) = p_{r0}(r) + \varepsilon \bar{p}_r(t, r), \quad (3.22)$$

$$p_\perp(t, r) = p_{\perp 0}(r) + \varepsilon \bar{p}_\perp(t, r), \quad (3.23)$$

$$m(t, r) = m_0(r) + \varepsilon \bar{m}(t, r), \quad (3.24)$$

$$R(t, r) = R_0(r) + \varepsilon D(t)e_1(r), \quad (3.25)$$

$$T(t, r) = T_0(r) + \varepsilon D(t)e_2(r), \quad (3.26)$$

$$f(R, T) = [R_0(r) + \alpha R_0^2(r) + \lambda T_0] + \varepsilon D(t)e_1(r)[1 + 2\alpha R_0(r)] + \eta D(t)e_2(r), \quad (3.27)$$

$$f_R = 1 + 2\alpha R_0(r) + \varepsilon 2\alpha D(t)e_1(r), \quad (3.28)$$

$$f_T = \lambda, \quad (3.29)$$

$$\Theta(t, r) = \varepsilon \bar{\Theta}. \quad (3.30)$$

Without loss of generality, we have taken the Schwarzschild coordinate $Y_0(r) = r$ and apply perturbation scheme on dynamical equations i.e., Eqs.(2.15) and (2.16), the perturbed Bianchi identities turn out to be

$$\begin{aligned} & \dot{\bar{\rho}} + \left[\frac{2e\rho_0}{1 + 2\alpha R_0} + \lambda_1 \left\{ \frac{2\bar{y}}{r}(\rho_0 + 2p_{\perp 0}) + \frac{x}{X_0}(\rho_0 + p_{r0}) \right\} \right. \\ & \left. + (1 + 2\alpha R_0)Z_{1p} \right] \dot{D} = 0, \\ & \lambda_1 \left\{ \bar{p}_r' + \bar{\rho} \frac{W_0'}{W_0} + \bar{p}_r \left(\frac{W_0'}{W_0} + \frac{2}{r} - \frac{2\alpha R_0'}{1 + 2\alpha R_0} \right) - \frac{2\bar{p}_\perp}{r} \right\} + 2\alpha \left[\frac{1}{W_0^2} (e' \right. \\ & \left. + 2e \frac{X_0'}{X_0} - \frac{x}{X_0} R_0') + X_0^2 (1 + 2\alpha R_0) \left\{ \frac{e}{X_0^2 (1 + 2\alpha R_0)} \right\}' \right] \ddot{D} + D [\lambda_1 [(\rho_0 \end{aligned} \quad (3.31)$$

$$\begin{aligned}
& +p_{r0})\left(\frac{w}{W_0}\right)' - 2(p_{r0} + p_{\perp 0})\left(\frac{\bar{y}}{r}\right)' + \lambda\bar{\rho}' - \frac{2\alpha}{1+2\alpha R_0} \left\{ \lambda_1 \left(p'_{r0} + \rho_0 \frac{W'_0}{W_0} \right. \right. \\
& \left. \left. + p_{r0} \left(\frac{2}{r} + \frac{W'_0}{W_0} - \frac{2\alpha R'_0}{1+2\alpha R_0} \right) \right) \right\} + \lambda \left(e' + e[\rho'_0 - \frac{2\alpha R'_0}{1+2\alpha R_0}] \right) \\
& + (1+2\alpha R_0)Z_{2p}] = 0.
\end{aligned} \tag{3.32}$$

For the sake of simplicity we assume that $e_1 = e_2 = e$ and set $\lambda_1 = \lambda + 1$, Z_{1p} and Z_{2p} are provided in appendix.

Elimination of $\dot{\bar{\rho}}$ from Eq. (3.31) and integration of resultant with respect to time yields an expression for $\bar{\rho}$ of following form

$$\bar{\rho} = - \left[\frac{2e\rho_0}{1+2\alpha R_0} + \lambda_1 \left\{ \frac{2\bar{y}}{r}(\rho_0 + 2p_{\perp 0}) + \frac{x}{X_0}(\rho_0 + p_{r0}) \right\} + (1+2\alpha R_0)Z_{1p} \right] D. \tag{3.33}$$

The perturbed on field equation Eq.(2.11) leads to the expression for \bar{p}_{\perp} , that turns out to be

$$\begin{aligned}
\bar{p}_{\perp} = & \frac{\ddot{D}}{W_0^2} \left\{ \frac{(1+2\alpha R_0)\bar{y}}{r} - 2\alpha e \right\} - \frac{\lambda\bar{\rho}}{\lambda_1} + \left\{ \left(p_{\perp 0} - \frac{\lambda}{\lambda_1}\rho_0 \right) \frac{2\alpha e}{1+2\alpha R_0} \right. \\
& \left. + \frac{Z_3}{\lambda_1} \right\} D,
\end{aligned} \tag{3.34}$$

effective part of the field equation is denoted by Z_3 , given in appendix Eq.(6.5). Matching conditions at boundary surface reveals

$$p_r \stackrel{\Sigma}{=} 0, \quad p_{\perp} \stackrel{\Sigma}{=} 0 \tag{3.35}$$

Above equation together with perturbed form of Eq.(2.11) can be written in the following form

$$\ddot{D}(t) - Z_4(r)D(t) = 0, \tag{3.36}$$

provided that

$$\begin{aligned}
Z_4 = & \frac{rW_0^2}{(1+2\alpha R_0)\bar{y} - 2\alpha er} \left[\frac{2\alpha e}{1+2\alpha R_0} p_{\perp 0} + \lambda \left\{ \frac{2\bar{y}}{r}(\rho_0 + 2p_{\perp 0}) \right. \right. \\
& \left. \left. + \frac{x}{X_0}(\rho_0 + p_{r0}) \right\} + (1+2\alpha R_0)Z_{1p} + \frac{Z_3}{\lambda_1} \right].
\end{aligned} \tag{3.37}$$

The solution of Eq.(3.36) takes form

$$D(t) = -e^{\sqrt{Z_4}t}. \tag{3.38}$$

The terms appearing in Z_4 are presumed in a way that all terms remain positive to have a valid solution for D . Expansion-free condition and stability range is discussed in the following section.

4 Expansion-free Condition with Newtonian and Post-Newtonian Limits

The models with an additional zero expansion condition delimitate two hypersurfaces, one separates the external Schwarzschild solution from the fluid distribution and other is the boundary between fluid distribution and internal cavity. Such models have extensive astrophysical applications where cavity within fluid distribution exists and are significant in investigation of voids at cosmological scales [49]. The spongelike structures are termed as voids existing in different sizes i.e., mini-voids to super-voids [50, 51] accompanying almost 50% of the universe, considered as vacuum spherical cavities within fluid distribution.

Implementation of linear perturbation on Eq.(2.7) and Eq.(2.12) respectively, implies

$$\bar{\Theta} = \frac{\dot{D}}{W_0} \left(\frac{x}{X_0} + 2\frac{\bar{y}}{r} \right)' \quad (4.39)$$

$$m_0 = \frac{r}{2} \left(1 - \frac{1}{X_0^2} \right), \quad (4.40)$$

$$\bar{m} = -\frac{D}{X_0^2} \left[r \left(\bar{y}' - \frac{x}{X_0} \right) + (1 - X_0^2) \frac{\bar{y}}{2} \right]. \quad (4.41)$$

The expansion free condition implies vanishing expansion scalar i.e., $\Theta = 0$ implying

$$\frac{x}{X_0} = -2\frac{\bar{y}}{r}. \quad (4.42)$$

In order to present the dynamical analysis in Newtonian (N) and post-Newtonian (pN) limits, we assume

$$\rho_0 \gg p_{r0}, \quad \rho_0 \gg p_{\perp 0}. \quad (4.43)$$

The metric coefficients upto the pN approximation in c.g.s. units are taken as

$$W_0 = 1 - \frac{Gm_0}{c^2 r}, \quad X_0 = 1 + \frac{Gm_0}{c^2 r}, \quad (4.44)$$

where c denote speed of light and G stands for gravitational constant. Expression for $\frac{X'_0}{X_0}$ can be obtained from Eq. (4.40) as

$$\frac{X'_0}{X_0} = \frac{-m_0}{r(r-2m_0)}, \quad (4.45)$$

Eq. (4.40) together with (2.10) implies

$$\begin{aligned} \frac{W'_0}{W_0} = & \frac{1}{2r(r-2m_0)(1+2\alpha R_0+r\alpha R'_0)} \left[r^3(\lambda_1 p_{r0} + \lambda \rho_0 - R_0 \right. \\ & \left. - 3\alpha R_0^2) + 2\alpha r(R_0 - 2rR'_0 + 4R'_0 m_0) + 2m_0 \right]. \end{aligned} \quad (4.46)$$

The static configuration of first Bianchi identity is identically satisfied while second provides a fruitful result in terms of dynamical equation. Substitution of Eqs. (4.45) and (4.46) in statically configured Eq. (2.16) and after some manipulation the dynamical equation in relativistic units yield

$$\begin{aligned} p'_{r0} = & - \left[\frac{\lambda}{\lambda_1} \rho'_0 + \frac{r(1+2\alpha R_0)}{r-2m_0} \left[\frac{r-2m_0}{r(1+2\alpha R_0)} \left\{ \frac{\alpha R_0^2}{2} - \frac{2\alpha R'_0(r-2m_0)}{r} \left(\frac{2}{r} \right. \right. \right. \right. \\ & + \frac{1}{2r(r-2m_0)(1+2\alpha R_0+r\alpha R'_0)} \left[r^3(\lambda_1 p_{r0} + \lambda \rho_0 - R_0 - 3\alpha R_0^2) \right. \\ & \left. \left. \left. + 2\alpha r(R_0 - 2rR'_0 + 4R'_0 m_0) + 2m_0 \right] \right\} \right]_{,1} + \frac{2m_0}{r(r-2m_0)} (p_{r0} - p_{\perp 0} \\ & + \frac{\alpha R_0^2}{4} - \frac{3}{r}) + \frac{2}{r^2} - \frac{2\alpha R'_0}{1+2\alpha R_0} \left(p_{r0} + \frac{\lambda}{\lambda_1} \right) + \frac{2\alpha R''_0(r-2m_0)}{r^2} \\ & + \frac{1}{2r(r-2m_0)(1+2\alpha R_0+r\alpha R'_0)} \left[r^3(\lambda_1 p_{r0} + \lambda \rho_0 - R_0 - 3\alpha R_0^2) \right. \\ & \left. + 2\alpha r(R_0 - 2rR'_0 + 4R'_0 m_0) + 2m_0 \right] \left[\rho_0 + p_{r0} + \frac{2\alpha R'_0(r-2m_0)}{r} \right. \\ & \left. \times \left\{ \frac{3m_0}{r(r-2m_0)} - \frac{1}{2r(r-2m_0)(1+2\alpha R_0+r\alpha R'_0)} \left[r^3(\lambda_1 p_{r0} + \lambda \rho_0 \right. \right. \right. \\ & \left. \left. \left. - 3\alpha R_0^2) + 2\alpha r(R_0 - 2rR'_0 + 4R'_0 m_0) + 2m_0 - R_0 \right] \right\} \right]. \end{aligned} \quad (4.47)$$

In c.g.s. units, we may write above equation as

$$\begin{aligned}
p'_{r0} = & - \left[\frac{c^{-2}r(1+2\alpha R_0)}{r-2Gc^{-2}m_0} \left[\frac{r-2Gc^{-2}m_0}{rc^{-2}(1+2\alpha R_0)} \left\{ -\frac{2\alpha R'_0(r-2Gc^{-2}m_0)}{rc^{-2}} \left(\frac{2}{r} \right. \right. \right. \right. \\
& + \frac{1}{2r(r-2c^{-2}m_0)(1+2\alpha R_0+r\alpha R'_0)} [r^3(\lambda_1 c^{-2}p_{r0} + \lambda\rho_0 - R_0 \\
& - 3\alpha R_0^2) + 2\alpha r(R_0 - 2rR'_0 + 4R'_0 Gc^{-2}m_0) + 2Gc^{-2}m_0] + \frac{\alpha R_0^2}{2} \left. \left. \left. \left. \right\} \right] \right]_{,1} \\
& + \frac{2Gc^{-2}m_0}{rc^{-2}(r-2Gc^{-2}m_0)} \left(p_{r0} - p_{\perp 0} + \frac{\alpha R_0^2}{4} - \frac{3}{r} \right) - \frac{2\alpha R'_0}{1+2\alpha R_0} (c^{-2}p_{r0} \\
& + \frac{\lambda}{\lambda_1}) + \frac{2}{c^{-2}r^2} + \frac{1}{2r(r-2Gc^{-2}m_0)(1+2\alpha R_0+r\alpha R'_0)} [r^3(\lambda_1 c^{-2}p_{r0} \\
& + \lambda\rho_0 - R_0 - 3\alpha R_0^2) + 2\alpha r(2Gc^{-2}m_0 - 2rc^{-2}R'_0 + 4R'_0 Gc^{-2}m_0) \\
& + R_0] \left[\rho_0 + p_{r0} + \left\{ \frac{-1}{2r(r-2Gc^{-2}m_0)(1+2\alpha R_0+rc^{-2}\alpha R'_0)} [r^3(\lambda\rho_0 \right. \right. \\
& + \lambda_1 c^{-2}p_{r0} - 3\alpha R_0^2) + 2\alpha rc^{-2}(R_0 - 2rR'_0 + 4R'_0 Gc^{-2}m_0) - R_0 \\
& + 2Gc^{-2}m_0] + \frac{3Gc^{-2}m_0}{r(r-2Gc^{-2}m_0)} \left. \left. \right\} \frac{2\alpha R'_0(r-2Gc^{-2}m_0)}{rc^{-2}} \right] + \frac{\lambda}{\lambda_1} \rho'_0 \\
& + \frac{2\alpha R''_0(r-2Gc^{-2}m_0)}{r^2} \left. \right]. \tag{4.48}
\end{aligned}$$

The terms of order c^0 and c^{-2} belongs to N and pN-approximation respectively. One can expand Eq. (4.48) upto c^{-2} and separate the terms of N and pN limits to distinguish the physical quantities lying in various regimes.

The use of expansion-free condition in Eq. (3.33) modifies $\bar{\rho}$ to following form

$$\bar{\rho} = - \left[\frac{2e\rho_0}{1+2\alpha R_0} + \lambda_1 \frac{2\bar{y}}{r} ((p_{r0} - p_{\perp 0})) + (1+2\alpha R_0)Z_{1p} \right] D. \tag{4.49}$$

The Harrison-Wheeler type equation of state describing second law of thermodynamics relates $\bar{\rho}$ and \bar{p}_r in terms of adiabatic index Γ as

$$\bar{p}_r = \Gamma \frac{p_{r0}}{\rho_0 + p_{r0}} \bar{\rho}. \tag{4.50}$$

Γ measures the fluid's compressibility belonging to its stiffness. Inserting $\bar{\rho}$

from Eq.(4.49) in Eq.(4.50), we have

$$\bar{p}_r = -\Gamma \frac{p_{r0}}{\rho_0 + p_{r0}} \left[\frac{2e\rho_0}{1 + 2\alpha R_0} + \lambda_1 \frac{2\bar{y}}{r} ((p_{r0} - p_{\perp 0})) + (1 + 2\alpha R_0) Z_{1p} \right] D. \quad (4.51)$$

In view of dimensional analysis, it is found that terms of \bar{p}_r and $\rho_0 \frac{W'_0}{W_0}$ lies in post post Newtonian (ppN) era and thus can be ignored from the terms of N and pN approximation. Since we are going to exclude \bar{p}_r , so it is intuitively clear that instability range is independent of Γ , no compression is introduced. Use of expansion-free condition together with the expression found for D , it follows that

$$\begin{aligned} & 2\alpha Z_4 \left[\frac{1}{W_0^2} \left(e' + 2e \frac{X'_0}{X_0} + 2 \frac{\bar{y}}{r} R'_0 \right) + X_0^2 (1 + 2\alpha R_0) \left\{ \frac{e}{X_0^2 (1 + 2\alpha R_0)} \right\}' \right] \\ & + \lambda_1 \left[\rho_0 \left(\frac{w}{W_0} \right)' - 2(p_{r0} + p_{\perp 0}) \left(\frac{\bar{y}}{r} \right)' \right] + \lambda \bar{\rho}' - \frac{2\alpha}{1 + 2\alpha R_0} \left\{ \lambda_1 \left(p'_{r0} + p_{r0} \left(\frac{2}{r} \right. \right. \right. \\ & \left. \left. \left. - \frac{2\alpha R'_0}{1 + 2\alpha R_0} \right) \right) \right\} + \lambda \left(e' + e[\rho'_0 - \frac{2\alpha R'_0}{1 + 2\alpha R_0}] \right) + (1 + 2\alpha R_0) Z_{2p} = 0, \end{aligned} \quad (4.52)$$

For simplification of above expression, we take relativistic units and assume that $\rho_0 \gg p_{r0}$, $\rho_0 \gg p_{\perp 0}$. On substitution of expressions for Z_4 , W_0 , X_0 , $\frac{X'_0}{X_0}$ and Z_{2p} respectively from Eqs. (3.37), (4.44), (4.45) and (6.4) yields a very lengthy expression defining the factors affecting the instability range at N and pN limits. The expanded version of Eq. (4.52) is large enough, therefore we are quoting only the results obtained from the collapse equation together with the restrictions to be imposed on physical parameters. It is clear from Eq. (4.47) that $p'_{r0} < 0$, provided that all the terms maintain positivity to fulfill stability criterion. Negative values of p'_{r0} depicts decrease in pressure with time transition, leading to collapse of gravitating star. Furthermore, using the c.g.s. units, it is found that the terms of \bar{p}_r and $\rho_0 \frac{W'_0}{W_0}$ does not take part in evolution for N and pN approximations since these terms belong to ppN limit. The analysis of terms lying in N and pN limits imply few restrictions to be imposed on physical quantities for discussion of instability range. These are listed as follows:

- **Newtonian Regime:** The constraints on material parameters are

$$p_{r0} > p_{\perp 0}, \quad \alpha^2 r R'_0 < 1 + 2\alpha R_0, \quad \frac{2\alpha R'_0}{1 + 2\alpha R_0} > \rho'_0 - e'.$$

The gravitating body remain unstable as long as the inequalities hold in N approximation.

- **post-Newtonian Regime:** In pN limits following restrictions are found to execute the instability range

$$p_{r0} > p_{\perp 0}, \quad r > 2m_0, \quad \frac{r}{r+m_0} \left(xR'_0 + \frac{2em_0}{r} \right) < e' + \lambda,$$

$$2\alpha e - (1 + 2\alpha R_0) \frac{\bar{y}}{r} > \frac{(r^2 - m_0^2)^2}{r^4} \left\{ \frac{er^2}{(r+m_0)} \right\}', \quad (r - 2m_0) > \frac{R'_0}{2R_0}.$$

5 Summary and Results

The mysterious content named as dark energy (DE) occupying the major part of universe is significant in the description of cosmic speed-up. The modified gravity theories are assumed to be effective in understanding cosmic acceleration by induction of so-called dark matter components in the form of higher order curvature invariants. Among such theories, $f(R, T)$ represents non-minimal coupling of matter and geometry. It provides an alternate to incorporate dark energy components and cosmic acceleration [52]. Thus consideration of $f(R, T)$ for dynamical analysis is worthwhile, covering the impact of higher order curvature terms and trace of energy momentum tensor T . This manuscript is based on the role of viable $f(R, T)$ model in establishment of instability range of spherically symmetric star.

Our exploration regarding viability of the $f(R, T)$ model reveals that the selection of $f(R, T)$ model for dynamical analysis is constrained to the form $f(R, T) = f(R) + \lambda T$, where λ is arbitrary positive constant. The restriction on $f(R, T)$ form originates from the complexities of non-linear terms of trace in analytical formulation of field equations. The $f(R, T)$ form we have chosen mainly is $f(R, T) = R + \alpha R^2 + \lambda T$, in agreement with the stable stellar configuration and satisfies the cosmic viability. The matter configuration is assumed to have unequal stresses i.e., anisotropic with central vacuum cavity evolving under expansion-free condition. Zero expansion condition on anisotropic background reveals the significance of energy density profile and pressure inhomogeneity in structure formation and evolution.

The field equations framed in $f(R, T)$ gravity are formulated and their conservation is considered to study the evolution. Conservation laws yield dynamical equations that are significant in formation of collapse equation.

In order to examine the variation from static equilibrium, we introduced linear perturbation to all physical parameters. The expressions for perturbed configuration of field equations reveal expressions for energy density $\bar{\rho}$ and tangential pressure \bar{p}_\perp . The second law of thermodynamics relating radial pressure and density with the help of adiabatic index Γ is considered to extract \bar{p}_r .

On account of zero expansion it is found that fluid evolution is independent of Γ , rather instability range depends on higher order curvature corrections and static pressure anisotropy. Recently, the dynamical analysis of isotropic and anisotropic spherical stars in $f(R, T)$ has been studied in [39, 40]. It is found that perturbed form of dark source terms of collapse equation also has the contribution of trace T , affecting the stability range. Thus non-minimal coupling of the higher order curvature terms and trace of energy momentum tensor imply a wider range of stability, however, the fluid evolving with zero expansion might cause drastic and unexpected variations. As expansion-free condition produces shear blow-up in gravitating system, so it is very captivating to extend this work for shearing expansion free case. The results are in accordance with [43] for vanishing λ , for vanishing α and λ corrections to GR solution can be found.

In addition to model (3.17), nature of various $f(R, T)$ models i.e., $f(R, T) = R + \alpha R^n + \lambda T$, $f(R, T) = R + \alpha R^2 + \frac{\mu^4}{R} + \lambda T$ and $f(R, T) = R + \frac{\mu^4}{R} + \lambda T$ has been briefly discussed in this section, as follows:

- $f(R, T) = R + \alpha R^n + \lambda T$: The model $f(R, T) = R + \alpha R^n + \lambda T$ is viable for any $n \geq 2$ and positive constants α and λ . The collapse equation for such model with zero expansion is of the form

$$\begin{aligned} & \lambda \bar{\rho}' + \lambda_1 \left\{ \bar{p}_r' + (\bar{\rho} + \bar{p}_r) \frac{W_0'}{W_0} + \bar{p}_r \left(\frac{2}{r} - \frac{\alpha n(n-1)R_0^{n-2}R_0'}{1 + \alpha n R_0^{n-1}} \right) - \frac{2}{r} \bar{p}_\perp \right\} \\ & + D \left[\lambda_1(\rho_0 + p_{r0}) \frac{w'}{W_0} + \lambda_1 \frac{2}{r} (p_{r0} - p_{\perp 0}) - (\lambda + \lambda_1 p_{r0}) \frac{\alpha n(n-1)(R_0^{n-2}e)'}{1 + \alpha n R_0^{n-1}} \right] \\ & + Z_{3p}, \end{aligned} \quad (5.53)$$

where pressure stresses \bar{p}_r, \bar{p}_\perp can be generated from perturbed field equations and perturbed energy density is

$$\bar{\rho} = - \left[\frac{\alpha n(n-1)(R_0^{n-2}e)'}{1 + \alpha n R_0^{n-1}} \rho_0 + \lambda_1 (p_{r0} - p_{\perp 0}) \frac{x}{X_0} + Z_{4p} \right] D, \quad (5.54)$$

where Z_{3p} and Z_{4p} depict the perturbed dark source terms. The N and pN limit of this model reveals that the term \bar{p}_r and $\rho_0 \frac{W'_0}{W_0}$ belong to ppN limit and so do not contribute in evolution. In Newtonian limit the physical quantities must satisfy the following conditions

$$p_{r0} > p_{\perp 0}, \quad \alpha^2 r n(n-1) R_0^{n-2} R'_0 < 1 + n\alpha R_0^{n-1},$$

$$\frac{n\alpha n(n-1) R_0^{n-2} R'_0}{1 + n\alpha R_0^{n-1}} > \rho'_0 - e'.$$

The constraints on physical quantities in pN regime are

$$r > 2m_0, \quad \frac{r}{r+m_0} (xn(n-1) R_0^{n-1} R'_0 + \frac{2em_0}{r}) < e' + \lambda,$$

$$2\alpha e - (1 + n\alpha R_0^{n-1}) \frac{\bar{y}}{r} > \frac{(r^2 - m_0^2)^2}{r^4} \left\{ \frac{er^2}{(r+m_0)} \right\}',$$

$$(r - 2m_0) > \frac{(n-1) R_0^{n-1} R'_0}{R_0^{n-1}}, p_{r0} > p_{\perp 0}.$$

The dynamical analysis of various models involving higher order curvature terms, combined with the trace of energy momentum tensor can be presented for $n \geq 2$.

- $f(R, T) = R + \alpha R^2 + \frac{\mu^4}{R} + \lambda T$ **and** $f(R, T) = R + \frac{\mu^4}{R} + \lambda T$: The perturbed form of Bianchi identity for $f(R, T) = R + \alpha R^2 + \frac{\mu^4}{R} + \lambda T$, where μ is arbitrary constant leads to the expression for $\bar{\rho}$ as follows

$$\bar{\rho} = - \left[\frac{2e\rho_0}{1 + 2\alpha R_0 - \mu^4 R_0^{-2}} + \lambda_1 \left\{ \frac{x}{X_0} (p_{r0} - p_{\perp 0}) \right\} + (1 + 2\alpha R_0 - \mu^4 R_0^{-2}) Z_{1p} \right] D. \quad (5.55)$$

The evolution equation becomes

$$\lambda_1 \left\{ \bar{p}_r' + \bar{\rho} \frac{W'_0}{W_0} + \bar{p}_r \left(\frac{W'_0}{W_0} + \frac{2}{r} - 2 \frac{\alpha R'_0 + \mu^4 R_0^{-3} R'_0}{1 + 2\alpha R_0 - \mu^4 R_0^{-2}} \right) - \frac{2\bar{p}_{\perp}}{r} \right\} + D [\lambda_1 [(\rho_0$$

$$+ p_{r0}) \left(\frac{w}{W_0} \right)' - 2(p_{r0} + p_{\perp 0}) \left(\frac{\bar{y}}{r} \right)'] + \lambda \bar{\rho}' - \frac{2\alpha + \mu^4 R_0^{-3}}{1 + 2\alpha R_0 - \mu^4 R_0^{-2}} \left\{ \lambda_1 \left(p'_{r0} + \rho_0 \frac{W'_0}{W_0} \right. \right.$$

$$+ p_{r0} \left(\frac{2}{r} + \frac{W'_0}{W_0} - 2 \frac{\alpha R'_0 + \mu^4 R_0^{-3} R'_0}{1 + 2\alpha R_0 - \mu^4 R_0^{-2}} \right) \left. \right\} + \lambda \left(e' + e[\rho'_0 - \frac{2\alpha R'_0}{1 + 2\alpha R_0}] \right)$$

$$+ (1 + 2\alpha R_0 - \mu^4 R_0^{-2}) Z_{5p}] = 0. \quad (5.56)$$

Z_{5p} denote the perturbed dark source entries. The N and pN limits are obtained by avoiding the terms lying in ppN region. To maintain the viability of model in Newtonian era, following inequalities must hold

$$\alpha R'_0 + \mu^4 R_0^{-3} R'_0 < 1 + 2\alpha R_0 - \mu^4 R_0^{-2}, \quad \frac{\alpha R'_0 + 1 + 2\alpha R_0 - \mu^4 R_0^{-2} R'_0}{1 + 2\alpha R_0 - \mu^4 R_0^{-2}} > \rho'_0 - e'.$$

In pN regime the system remains stable as long as following ordering relations are satisfied.

$$\begin{aligned} \frac{r}{r + m_0} (\alpha R'_0 + \mu^4 R_0^{-3} R'_0) &< e' + \lambda T_0, \\ 2\alpha e - \mu^4 R_0^{-2} - (1 + 2\alpha R_0 - \mu^4 R_0^{-2}) \frac{\bar{y}}{r} &> \frac{(r^2 - 2m_0^2)^2}{r^4} \left\{ \frac{er^2}{(r + m_0)} \right\}'. \end{aligned}$$

The collapse equation for $f(R, T) = R + \frac{\mu^4}{R} + \lambda T$ can be obtained by setting $\alpha = 0$ in Eq. (5.56), likewise the restrictions on physical quantities can be found.

6 Appendix

$$\begin{aligned}
Z_1(r, t) = & \left[\left\{ \frac{1}{f_R W^2} \left(\frac{f - R f_R}{2} - \frac{\dot{f}_R}{W^2} \Theta - \frac{f'_R}{X^2} \left(\frac{X'}{X} - \frac{2Y'}{Y} \right) + \frac{f''_R}{X^2} \right) \right\}_{,0} \right. \\
& + \left. \left\{ \frac{1}{f_R W^2 X^2} \left(\dot{f}'_R - \frac{W'}{W} \dot{f}_R - \frac{\dot{X}}{X} f'_R \right) \right\}_{,1} \right] f_R W^2 - \left\{ \left(\frac{\dot{X}}{X} \right)^2 \right. \\
& + 2 \left(\frac{\dot{Y}}{Y} \right)^2 + \frac{3\dot{W}}{W} \Theta \left. \right\} \frac{\dot{f}_R}{W^2} + \frac{\ddot{f}_R}{W^2} \Theta - \frac{2f'_R}{X^2} \left\{ \frac{\dot{W}}{W} \left(\frac{X'}{X} - \frac{Y'}{Y} \right) \right. \\
& + \frac{\dot{X}}{X} \left(\frac{2W'}{W} + \frac{X'}{X} + \frac{Y'}{Y} \right) + \frac{\dot{Y}}{Y} \left(\frac{W'}{W} - \frac{3Y'}{Y} \right) \left. \right\} + \frac{\dot{W}}{W} (f - R f_R) \\
& + \frac{f''_R}{X^2} \left(\frac{2\dot{W}}{W} + \frac{\dot{X}}{X} \right) + \frac{1}{X^2} \left(\dot{f}'_R - \frac{W'}{W} \dot{f}_R \right) \left(\frac{3W'}{W} + \frac{X'}{X} + \frac{2Y'}{Y} \right), \tag{6.1}
\end{aligned}$$

$$\begin{aligned}
Z_2(r, t) = & \left[\left\{ \frac{1}{f_R W^2 X^2} \left(\dot{f}'_R - \frac{W'}{W} \dot{f}_R - \frac{\dot{X}}{X} f'_R \right) \right\}_{,0} + \left\{ \frac{1}{f_R X^2} \left(\frac{R f_R - f}{2} \right. \right. \right. \\
& - \left. \left. \frac{\dot{f}_R}{W^2} \left(\frac{\dot{W}}{W} - \frac{2\dot{Y}}{Y} \right) - \frac{f'_R}{X^2} \left(\frac{W'}{W} + \frac{2Y'}{Y} \right) + \frac{\ddot{f}_R}{W^2} \right) \right\}_{,1} \right] f_R X^2 - \\
& \frac{\dot{f}_R}{W^2} \left\{ \frac{W'}{W} \left(\frac{\dot{W}}{W} + \frac{\dot{X}}{X} \right) + \frac{X'}{X} \left(\frac{\dot{W}}{W} - \frac{2\dot{Y}}{Y} \right) + \frac{2Y'}{Y} \left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} \right) \right\} \\
& + (R f_R - f) \frac{X'}{X} - \frac{1}{W^2} \left(\frac{\dot{W}}{W} + \frac{3\dot{X}}{X} + \frac{2\dot{Y}}{Y} \right) \left(\frac{W'}{W} \dot{f}_R + \frac{\dot{X}}{X} f'_R \right. \\
& - \left. \dot{f}'_R \right) - \frac{f'_R}{X^2} \left\{ \frac{W'}{W} \left(\frac{W'}{W} + \frac{3X'}{X} \right) + \frac{2Y'}{Y} \left(\frac{3X'}{X} + \frac{Y'}{Y} \right) \right\} + \frac{\ddot{f}_R}{W^2} \\
& \times \left(\frac{W'}{W} + \frac{2X'}{X} \right) + \frac{f''_R}{X^2} \left(\frac{W'}{W} + \frac{2Y'}{Y} \right). \tag{6.2}
\end{aligned}$$

$$\begin{aligned}
Z_{1p} = & 2\alpha W_0^2 \left[\frac{1}{W_0^2 X_0^2 (1 + 2\alpha R_0)} \left\{ e' - e \frac{W'_0}{W_0} - \frac{x}{X_0} R'_0 \right\} \right]_{,1} + \frac{1}{1 + 2\alpha R_0} \left[e \right. \\
& - [\lambda T_0 - \alpha R_0^2] \left(\frac{w}{W_0} + \frac{e}{1 + 2\alpha R_0} \right) - \frac{2\alpha}{X_0^2} \left\{ \left(\frac{X'_0}{X_0} - \frac{2}{r} \right) \left(2R'_0 \left(\frac{w}{W_0} \right. \right. \right. \\
& \left. \left. \left. + \frac{x}{X_0} \right) - e' - \frac{2\alpha e}{1 + 2\alpha R_0} R'_0 \right) + R''_0 \left(\frac{2w}{W_0} + \frac{x}{X_0} \right) - 2R'_0 \left(\frac{x}{X_0} \left(\frac{2W'_0}{W_0} \right. \right. \right. \\
& \left. \left. \left. + \frac{X'_0}{X_0} + \frac{1}{r} \right) + \frac{\bar{y}}{r} \left(\frac{W'_0}{W_0} - \frac{3}{r} \right) \right) + \left(e' - e \frac{W'_0}{W_0} \right) \left(\frac{3W'_0}{W_0} + \frac{X'_0}{X_0} \right. \right. \\
& \left. \left. + \frac{2}{r} \right) \right\} \left. \right] \quad (6.3)
\end{aligned}$$

$$\begin{aligned}
Z_{2p} = & X_0^2 (1 + 2\alpha R_0) \left[\frac{1}{X_0^2 (1 + 2\alpha R_0)} \left\{ e + \frac{2\alpha}{X_0^2} \left\{ \left(\frac{W'_0}{W_0} + \frac{2}{r} \right) \left(\frac{2\alpha e}{1 + 2\alpha R_0} \right. \right. \right. \right. \\
& \left. \left. \left. + \frac{4x}{X_0} \right) R'_0 - e \right\} - R'_0 \left[\left(\frac{w}{W_0} \right)' + \left(\frac{\bar{y}}{r} \right)' \right] \right\} + [\lambda T_0 - \alpha R_0^2] \left(\frac{e}{1 + 2\alpha R_0} \right. \right. \\
& \left. \left. + \frac{x}{X_0} \right) \right\} \right]_{,1} + x X_0 (1 + 2\alpha R_0) \left[\frac{-1}{X_0^2 (1 + 2\alpha R_0)} \left\{ \frac{4\alpha R'_0}{X_0^2} \left(\frac{W'_0}{W_0} + \frac{2}{r} \right) \right. \right. \\
& \left. \left. + \alpha R_0^2 - \lambda T_0 \right\} \right]_{,1} + \frac{2\alpha}{X_0^2} \left[R''_0 \left\{ \left(\frac{w}{W_0} \right)' - 2 \left(\frac{W'_0}{W_0} + \frac{2}{r} \right) \left(\frac{e}{1 + 2\alpha R_0} \right. \right. \right. \right. \\
& \left. \left. \left. + \frac{x}{X_0} \right) + \left(\frac{\bar{y}}{r} \right)' \right\} - R'_0 \left\{ \frac{W'_0}{W_0} \left[\left(\frac{2w}{W_0} \right)' + \left(\frac{3x}{X_0} \right)' \right] + 3 \frac{X'_0}{X_0} \left[\left(\frac{x}{X_0} \right)' \right. \right. \right. \\
& \left. \left. \left. + 2 \left(\frac{\bar{y}}{r} \right)' \right] + \frac{2}{r} \left[\left(3 \frac{x}{X_0} \right)' + 2 \left(\frac{\bar{c}}{r} \right)' \right] \right\} + \left(\frac{2x}{X_0} R'_0 - e \right) \left\{ \left(\frac{W'_0}{W_0} \right)^2 \right. \right. \\
& \left. \left. + 3 \frac{X'_0}{X_0} \left(\frac{W'_0}{W_0} + \frac{2}{r} \right) + \frac{2}{r^2} \right\} \right] + e \frac{X'_0}{X_0} - [\lambda T_0 - \alpha R_0^2] \left(\frac{2e}{1 + 2\alpha R_0} \frac{X'_0}{X_0} \right. \\
& \left. \left. + \frac{x}{X_0} \right) \right] \quad (6.4)
\end{aligned}$$

$$\begin{aligned}
Z_3 = & \frac{1 + 2\alpha R_0}{X_0^2} \left[\frac{w''}{W_0} + \frac{\bar{y}''}{r} - \frac{W_0''}{W_0} \left(\frac{w}{W_0} + \frac{2x}{X_0} \right) + \frac{W_0'}{W_0} \left\{ \frac{2x}{X_0} \left(\frac{X_0'}{X_0} - \frac{1}{r} \right) \right. \right. \\
& + \left. \left(\frac{\bar{y}}{r} \right)' - \left(\frac{x}{X_0} \right)' \right\} + \frac{X_0'}{X_0} \left\{ \frac{2xX_0'}{rX_0} - \left(\frac{w}{W_0} \right)' - \left(\frac{\bar{y}}{r} \right)' \right\} + \left\{ \left(\frac{w}{W_0} \right)' \right. \\
& - \left. \left(\frac{x}{X_0} \right)' \right\} \frac{1}{r} \left. \right] - \frac{2\alpha e}{1 + 2\alpha R_0} \left\{ \frac{\lambda T_0 - \alpha R_0^2}{2} - \frac{2\alpha}{X_0^2} \left(R_0' \left(\frac{W_0'}{W_0} - \frac{X_0'}{X_0} \right. \right. \right. \\
& + \left. \left. \left. \frac{1}{r} \right) - R_0'' \right) \right\} - \frac{2\alpha}{X_0^2} \left\{ e'' + \frac{2x}{X_0} R_0'' + \left(\frac{W_0'}{W_0} - \frac{X_0'}{X_0} + \frac{1}{r} \right) \left(\frac{2x}{X_0} R_0' \right. \right. \\
& \left. \left. - e' \right) \right\} \tag{6.5}
\end{aligned}$$

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